Mathematical Physics: Individual Competition

(To complete this assignment, answer 3 out of the 4 problems.)

Problem 1

Consider Born-Infeld electrodynamics with Lagrangian $\mathcal{L} = b^2\left(1 - \sqrt{1 - \frac{E^2 - B^2}{b^2}}\right)$.

1. Find the electrostatic field of a point charge and show it has finite self-energy.
2. Derive the speed of light in a background field E_0.

Problem 2

Consider a 1D Ising model with interactions decaying as $J(r) \propto r^{-\alpha}$ where $\alpha \in (1, 2)$.

1. Compute the approximate free energy for $\alpha = \frac{3}{2}$.
2. Find the critical temperature $T_c(\alpha)$

Problem 3

Maxwell theory in various dimensions. In any spacetime dimension $d \geq 2$, one can define a theory of a massless vector field A_μ with the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

1. Show that this Lagrangian is invariant under gauge transformations

   $$A_\mu \to A_\mu + \partial_\mu f,$$

   where f is an arbitrary real function.

2. Explain how to quantize this theory in the Coulomb gauge $\nabla \cdot \vec{A} = 0$, and determine the number of polarization states.

3. Show that, for $d = 2$, the only solution of the classical equations of motion is F_{12} being a constant. If the spacetime is the Euclidean space $\mathbb{R}^2$, find a solution of the gauge field A_μ that is rotational invariant with non-zero $F_{12} = c$. It might be convenient to work with polar coordinates.

4. When the space-time is S^2, glue together the solution for the previous part on the northern and southern hemisphere via a gauge transformation along the equator. Show that $\int_{S^2} F$ has to be quantized for the solution to be consistent.

5. Prove that, for $d = 3$, this theory on $\mathbb{R}^{2,1}$ is equivalent to a theory of a free massless scalar. (Hint: the scalar φ would satisfy $F = \star d\varphi$. Show that such φ exist and satisfy the equation of motion $\Box\varphi = 0$. Furthermore, show that the equal-time commutation relation for φ is the standard one.)

Problem 4

General Lorentz Transformation:

1. Please write down the Lorentz transformation of the moving system (t', x', y', z') with relative velocity v along the positive X-axis.

2. Now, could you use the result in part (a) to derive the General Lorentz Transformation when the relative motion is along any $\overrightarrow{\mathbf{v}}$ direction. The result should be in the form of vectors $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{r}}$ (not in the component form as x, y, z).

3. Now we consider the Lorentz transformation where the moving system has the relative velocity v

$$\overrightarrow{\mathbf{v}} = (v_x, v_y, 0)$$

Student A has tried to solve this problem by first taking the Lorentz transformation with relative velocity $(v_x, 0, 0)$ and then another Lorentz transformation with relative velocity $(0, v_y, 0)$. Student B has tried to solve this problem by first taking the Lorentz transformation with relative velocity $(0, v_y, 0)$ and then another Lorentz transformation with relative velocity $(v_x, 0, 0)$. Now the question is which student gives the right answer?